2050A Revision Exercise: 2017 1st term

1. Use the ε-$\mathbb{N}$ definition to show that $\lim \frac{n+(-1)^n}{n^2-1} = 0$.

2. Use the ε-$\mathbb{N}$ definition to show that $\lim_n \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n}\right) = 1$.

3. Using the definition show that the sequence $(\frac{n^2+1}{2n+1})$ diverges to ∞.

4. Show that if $x_n > 0$ and $\lim x_n = a$, then $\sqrt{x_n} \to \sqrt{a}$.

5. Suppose that $y_1 > x_1 > 0$ and $x_{n+1} = \sqrt{x_n y_n}$ and $y_{n+1} = \frac{x_n+y_n}{2}$. Show that $\lim x_n$ and $\lim y_n$ exist, moreover, $\lim x_n = \lim y_n$.

6. Show that if $\lim x_n = a$ exists, then $\lim \frac{x_1+\cdots+x_n}{n} = a$.

7. Show that if (x_n) is an unbounded sequence, then there is a subsequence (x_{n_k}) diverges to $+\infty$ or $-\infty$.

8. Suppose that (x_n) is an unbounded sequence and does not diverges to $+\infty$. Show that if (x_n) is bounded below, then there are two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that (x_{n_k}) diverges to $+\infty$ and $\lim_k x_{m_k}$ exists.

9. Suppose that $|r| < 1$ and (a_n) is bounded. Let $x_n := \sum_{k=0}^{n} a_k r^k$. Show that the sequence (x_n) is convergent.

10. Using the definition, show that $\lim_{x\to -1} \frac{x-3}{x^2-9} = \frac{1}{2}$; $\lim_{x\to\infty} \frac{x-1}{x+2} = 1$ and $\lim_{x\to\infty} \frac{x^2+x}{x+1} = \infty$.

11. Let $x \in [0,1]$ and $f(x) = 0$ if $x \in \mathbb{Q}$; otherwise, $f(x) = 1$. Find the right and left limits of f at $x = 1/2$.

12. Show that $\lim_{x\to\infty} f(x) = L$ exists if and only if for any sequence (x_n) with $x_n \to \infty$, we have $f(x_n) \to L$, where $L \in \mathbb{R}$ or $L = \infty$.

13. Let f be a function defined on $[a,b]$. Suppose that $\lim_{x\to c\pm} f(x)$ both exist for all $c \in [a,b]$. Show that f is bounded.

14. If f and g are continuous functions on $\mathbb{R}$, show that the function $h(x) := \max(f(x), g(x))$ for $x \in [a,b]$ is also continuous.

15. Let f be a continuous function defined on $[a,b]$. Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be any partition on $[a,b]$. Show that there is $\xi \in [a,b]$ such that $f(\xi) = \frac{f(x_0)+\cdots+f(x_n)}{n+1}$.

16. Show that if f is a continuous strictly positive function on $[a,b]$, then $\frac{1}{f(x)}$ is also continuous on $[a,b]$.

17. Prove by the definition that the functions $f(x) = x^{1/3}$ is uniformly continuous on $[0,1]$ and $g(x) = \sin x^2$ is not uniformly continuous on $\mathbb{R}$.
    Proof: **Claim:** $g(x)$ is not uniform continuous on $\mathbb{R}$. In fact, for each positive integer n, let $x_n = \sqrt{\frac{\pi}{2}}(n + 1/n)$ and $y_n = \sqrt{\frac{\pi}{2}}n$. Then $\sin \frac{x_n^2-y_n^2}{2} = \sin \frac{\pi}{4}(2 + 1/n^2)$ and $|\cos \frac{x_n^2+y_n^2}{2}| = |\sin(\frac{\pi}{2}n^2 + \pi/(4n^2))|$. Thus if we take $n = 2k+1$ and $k \to \infty$, then
    $$\sin x_n^2 - \sin y_n^2 = 2|\cos \frac{x_n^2+y_n^2}{2}||\sin \frac{x_n^2-y_n^2}{2}| \to 1$$
    but $|x_n - y_n| \to 0$. Therefore, the function g is not uniformly continuous on $\mathbb{R}$. □

18. Is the function $f(x) = x^2$ uniformly continuous on $\mathbb{R}$?
    Proof: Using the similar argument as in question 17, the result follows by considering $x_n = n + 1/n$ and $y_n = n$. □

19. Is the function $f(x) = \frac{\sin x}{x}$ uniformly continuous on $(0, \pi)$?
Proof: Define a function F on $[0, \pi]$ by $F(0) = 1$; $F(\pi) = 0$ and $F(x) = f(x)$ for $x \in (0, \pi)$. Then F is continuous on $[0, \pi]$ and thus, F is uniformly continuous on $[0, \pi]$. This implies that f is uniformly continuous on $(0, \pi)$ since the restriction $F|(0, \pi) = f$. □

20. Let f be a continuous function defined on $[a, \infty)$. Show that if $\lim_{x \to \infty} f(x)$ exists, then f is uniformly continuous on $[a, \infty)$. Is the converse true?
Proof: Let $\varepsilon > 0$. Since $\lim_{x \to \infty} f(x)$ exists, then by Cauchy Theorem, there is $M > a$ such that $|f(x) - f(y)| < \varepsilon$ as $x, y \geq M$. On the other hand, since f is uniformly continuous on $[a, M]$, we can find $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ as $x, y \in [a, M]$ with $|x - y| < \delta$. Therefore, we have $|f(x) - f(y)| < \varepsilon$ as $x, y \in \mathbb{R}$ with $|x - y| < \delta$. The proof is finished. □

21. Show that if f is a uniformly continuous function defined on (a, b), then f is bounded.
Proof: Since f is uniformly continuous on $\mathbb{R}$, then there is $\delta > 0$ such that $|f(x) - f(y)| < 1$ as $x, y \in (a, b)$ with $|x - y| < \delta$. Now we take a partition $a = x_0 < x_1 < \cdots < x_n = b$ with $|x_k - x_{k-1}| < \delta$ for all $k = 1, ..., n$. If we let $M := \max(|f(x_1)| + 1, ..., |f(x_{n-1})| + 1)$, then $|f(x)| < M$ for all $x \in (a, b)$. The proof is finished. □